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Hirzebruch Surfaces: Degenerations, Related Braid Monodromy, Galois Covers

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Dedicated to F. Hirzebruch on the occasion of his 70th birthday.

Abstract. We describe various properties of Hirzebruch surfaces and related constructions: degenerations, braid monodromy, Galois covers and their Chern numbers.

§0. Introduction

Hirzebruch surfaces were first introduced in 1951, in the paper *"Über eine Klasse von einfach-zusammenhängenden komplexen Mannigfaltigkeiten"* (see [**H**]). This paper is the first title reprinted in Hirzebruch's *Gesammelte Abhandlungen* (published in 1987 on the occasion of his 60th birthday), it is the first part of his dissertation and his very first mathematical paper. Hirzebruch studied the family of surfaces Σ_n for $n \geq 0$ that are given by the equation $x_1 y_1^n = x_2 y_2^n$ in $\mathbb{CP}^2 \times \mathbb{CP}^1$. He proved that analytically, these surfaces are mutually non-isomorphic, whereas topologically, being S^2-bundles over S^2, they fall into only two homeomorphism classes, and furthermore, he proved that they are all birationally equivalent. These surfaces, called *Hirzebruch surfaces*, have played an important role in the theory of algebraic surfaces ever since. Let us recall the construction as it is usually stated nowadays. For $n = k$, the k-th Hirzebruch surface is the projectivization of the vector bundle $\mathcal{O}_{\mathbb{CP}^1}(k) \oplus \mathcal{O}_{\mathbb{CP}^1}$. It is usually denoted by F_k. (In fact, any $\mathbb{CP}^1$-bundle over $\mathbb{CP}^1$ is some F_k).

Let σ be a holomorphic section of $\mathcal{O}_{\mathbb{CP}^1}(k)$, and let $E_0 \subset F_k$ denote the image of the section $(\sigma, 1)$ of $\mathcal{O}_{\mathbb{CP}^1}(k) \oplus \mathcal{O}_{\mathbb{CP}^1}$. The curve E_0 is called a *zero section* of F_k. All zero sections are homologous and hence define a divisor class which is independent of choice of σ. Let C denote a fiber of F_k. The Picard group of F_k is generated by E_0 and C. It is elementary that $E_0^2 = k$, $C^2 = 0$ and $E_0 \cdot C = 1$.

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The surface F_0 is the quadric $\mathbb{CP}^1 \times \mathbb{CP}^1$, and F_1 is the blow-up of the plane $\mathbb{CP}^2$. For $k > 0$, the surface F_k contains a unique (irreducible) curve of negative self-intersection $-k$. This curve is a section of the bundle; it is denoted E_∞ and it is called the *negative section* or the *section at infinity*. We mention that it can be contracted to an isolated normal singularity, the resulting normal surface being the cone over the rational normal curve of degree k. Zero sections are always disjoint to E_∞. Schematically we describe F_k as in Fig. 0.1.

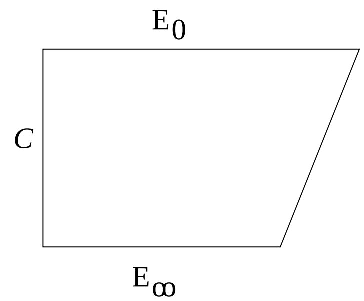


Fig. 0.1

This might be the place to point out that Hirzebruch, in his mathematical career, actually has studied many different classes of surfaces, apart from those that were named after him.

In the paper [**MoTe1**], published in the year of Hirzebruch's 60th birthday, we used the simplest of all Hirzebruch surfaces, namely, the quadric $F_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$, as the starting point to construct a simply connected surface of general type with a positive (topological) signature. That result disproved a famous conjecture in the theory of algebraic surfaces: The **Watershed conjecture** of Bogomolov (see [**FH**]) stated that a surface with non-negative signature should have an infinite fundamental group. The example was constructed as a Galois cover of F_0. To prove that it is simply connected, its fundamental group was determined by studying the braid monodromy of the branch curve corresponding to a generic projection from F_0, suitably embedded in some $\mathbb{CP}^N$, onto the plane $\mathbb{CP}^2$. This work was the starting point of a whole series of papers [**MoTe2**] – [**MoTe8**], and [**MoRoTe**], [**FRoTe**], [**Te1**] – [**Te4**], in which we present our algorithms for computing braid monodromy related to curves, degeneration of surfaces, fundamental groups of complements of curves, fundamental groups of Galois covers of surfaces, and Chern numbers of fibered products.

Some of the examples computed in these papers are based on Galois covers of Hirzebruch surfaces F_k. In addition to the counterexample, as in [**MoTe1**], we produced later the first examples of simply connected surfaces of general type with positive (topological) signature which are also spin manifolds ([**MoRoTe**]). Recall that the signature is positive if $c_1^2/c_2 > 2$. Corollary 6.3 of this paper gives such an example with $c_1^2/c_2 = 2.73$. We also computed an infinite series of pairs of surfaces with the same Chern numbers but with different fundamental group, where one group is trivial and the other of order going to infinity ([**RoTe**]).

We believe that fundamental groups of complements of branch curves can distinguish among surfaces lying in different connected components of moduli space.

One of our main tools is the braid group (and braid monodromy) technique as presented in [**MoTe3**] - [**MoTe6**]. The idea to use braid monodromy to compute fundamental groups of complements of curves started with Van Kampen and Enriques. Until the 1980's, very few works dealt with curves that occur as branch curves related to surfaces, in general, and to Hirzebruch surfaces, in particular. (See sections §3 and §4 below for such results). One can mention the works [**Za**] and [**Mo**]. It is important to note that the earlier works created a wrong impression about the complexity of these fundamental groups, namely, that they are "big", and in particular, that they contain free subgroups with two generators. The braid groups and their close analogues were considered as the typical examples. These expectations turned out to be false (see [**Te3**] for a list of examples). The results of Section 3 below are used in [**Te5**] for the precise computation of $\pi_1(\mathbb{CP}^2 \setminus S)$, where S is the branch curve of a generic projection of a Hirzebruch surface.

The paper is divided as follows:

§0. Introduction
§1. Construction and Degeneration of $F_{k(a,b)}$
§2. Braid Monodromy: Definition and Basic Properties
§3. Braid Monodromy Related to Generic Projection of $F_{k(a,b)}$
§4. Fundamental Groups of Galois Covers of Hirzebruch Surfaces
§5. Chern Numbers of Galois Covers
§6. Intermediate Galois Covers

§1. Construction and Degeneration of $F_{k(a,b)}$

Let F_k be the k-th Hirzebruch surface. Let E_0, E_∞, C be as in §0. For $a, b \geq 1$, or for $a = 0$ and $k \geq 1$, the divisor $aC + bE_0$ on F_k is very ample and thus defines an embedding $f_{|aC+bE_0|} : F_k \hookrightarrow \mathbb{CP}^N$. Let $F_{k(a,b)} = f_{|aC+bE_0|}(F_k)$ $(\subseteq \mathbb{CP}^N)$. For $k > 0$, the map $f_{|0 \cdot C+bE_0|}$ collapses the section at infinity to a point, so $F_{k(0,b)}$ is the image of the cone over the rational normal curve of degree k with respect to a suitable embedding.

In [**MoRoTe**] we constructed a degeneration to a union of $2ab + kb^2$ planes in the following configuration (in Fig. 1.1, we took $k = 2$, $a = 2$, $b = 3$). Each triangle represents a plane and each inner edge represents an intersection line between planes.

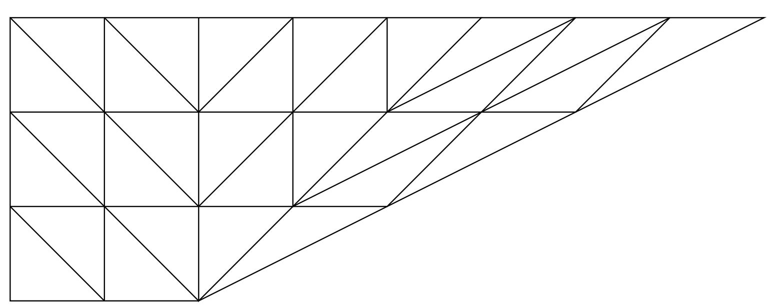


Fig. 1.1

This degeneration is obtained using a technique developed by us which we refer to as the D-construction. The D-construction is described (and proven to

work) in [**MoTe5**]. Specific degeneration for the Hirzebruch surfaces using the D-construction is explained in [**MoRoTe**], Section 2 (Theorem 2.1.2). The difference between the D-construction and other blow-up procedures for obtaining degenerations is that we can apply the D-construction also along a subvariety of codim 0 (see, for example, Step 2 below). The degeneration is obtained via the following steps:

1. D-construction along C to get $F_{0(1,b)} \cup F_{k(a-1,b)}$.
2. D-construction along $F_{0(1,b)}$ to get $F_{0(1,b)} \cup F_{0(1,b)} \cup F_{k(a-2,b)}$.
3. Induction on the second step to get $\underbrace{F_{0(1,b)} \cup \cdots \cup F_{0(1,b)}}_{a \text{ times}} \cup F_{k(0,b)}$ (see [**MoTe5**])
4. Degeneration of each $F_{0(1,b)}$ to a union of $2b$ planes in the following configuration (here $b = 3$).

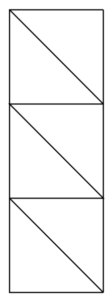


Fig. 1.2

5. D-construction on $F_{k(0,b)}$ to get $\underbrace{F_{1(0,b)} \cup \cdots \cup F_{1(0,b)}}_{k \text{ times}}$.

   ($F_{1(0,b)}$ is the Veronese surface V_b)
6. Degeneration of each $F_{1(0,b)}$ to a union of b^2 planes in the following configuration (here $b = 3$):

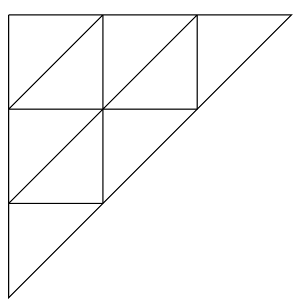


Fig. 1.3

Remarks.

1. We could go in the "reverse" direction of the degeneration and replace $\underbrace{F_{0(1,b)} \cup \cdots \cup F_{0(1,b)}}_{a \text{ times}}$ by $F_{0(a,b)}$. In other words, we might consider a degeneration of $F_{k(a,b)}$ to $F_{0(a,b)} \cup \bigcup_{k \text{ times}} V_b$.

2. There are other procedures in progress to obtain a degeneration of V_b to a union of planes.
3. In [**CiMiTe**] we shall use the above degeneration to describe a new degeneration of a K3-surface.

§2. Braid Monodromy: Definition and Basic Properties

In this section we present braid monodromy and braid monodromy factorizations in general, and in the next section we shall discuss the one related to Hirzebruch surfaces.

Throughout this section (and in section 4) we shall use the following notations:

S is a curve in $\mathbb{C}^2$ defined over the reals, $p = \deg S$.

$\pi : \mathbb{C}^2 \to \mathbb{C}$, $\pi(x, y) = x$, is the first coordinate projection, in a generic coordinate system defined over the reals.

$K(x) = \{y \mid (x, y) \in S\}$.

$N = \{x \mid \#K(x) \lneqq p\}$ (w.l.o.g. $N \subseteq \mathbb{R}$ since braid monodromy is defined up to homotopy type).

$M' = \{(x, y) \in S \mid \pi \text{ is not étale at } (x, y)\}$ (clearly, $\pi(M') = N$ and by genericity $\#(\pi^{-1}(x) \cap M') = 1,\ \forall x \in N$).

Let E (resp. D) be a closed disk on the x-axis (resp. y-axis) such that $M' \subset Int(E \times D)$, ($N \subset \text{Int}(E)$.)

We choose $u \in \partial E$, real, $x \ll u$, $\forall x \in N$. (Clearly, $\#(\pi^{-1}(u) \cap S) = p$.)

$K = K(u) = \{q_1, \ldots, q_p\}$.

In such a situation, we are going to introduce *braid monodromy.*

Definition. Braid monodromy of an affine curve S w.r.t. $E \times D, \pi, u$

Every loop in $E \setminus N$ starting at u has liftings to a system of p paths in $(E \setminus N) \times D) \cap S$ starting at $q_1, \ldots, q_p$. Projecting them horizontally to D, we get p paths $\{q_1(t), \ldots, q_p(t)\}$ in D, each one starts and ends in K, which together can be referred to as a motion.

This motion defines a braid in $B_p[D, K]$ (see [**MoTe3**], Section III). Thus we get a map $\varphi : \pi_1(E \setminus N, u) \to B_p[D, K]$. This map is evidently a group homomorphism, and it is *the braid monodromy of S w.r.t. $E \times D, \pi, u$*. We sometimes denote φ by φ_u.

It is better to have a notion of braid monodromy of a curve not depending on the choice of D and E, when possible and needed:

Definition. Braid monodromy of S w.r.t. π, u

Let $\mathbb{C}^1_u = \{(u, y) \mid y \in \mathbb{C}\}$. When considering the braid induced from the previous motion as an element of the group $B_p[\mathbb{C}_u, K]$ we get the homomorphism $\varphi : \pi_1(\mathbb{C} \setminus N, u) \to B_p[\mathbb{C}^1_u, K]$ which is called *the braid mondromy of S w.r.t. π, u.*

In order to present an example of a braid monodromy calculation, we recall a geometric model of the braid group and the definition of a half-twist.

Definition. Braid group $B_n[D, K]$

Let D be a closed disk in $\mathbb{R}^2$, $K \subset \text{Int}(D)$, K finite. Let B be the group of all diffeomorphisms β of D such that $\beta(K) = K$, $\beta|_{\partial D} = \text{Id}|_{\partial D}$. For $\beta_1, \beta_2 \in B$, we say that β_1 is equivalent to β_2 if β_1 and β_2 induce the same automorphism of $\pi_1(D \setminus K, u)$. The quotient of B by this equivalence relation is called *the braid group*

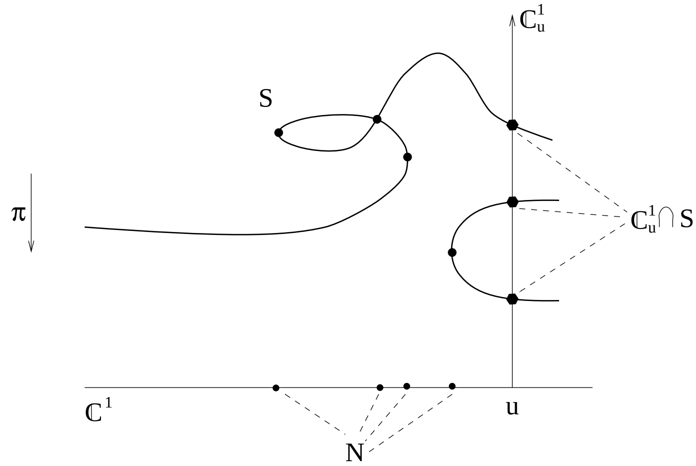


FIG. 2.1

$B_n[D,K]$ $(n = \#K)$. The elements of $B_n[D,K]$ are called *braids.* We sometimes denote by $\overline{\beta}$ the braid represented by β.

DEFINITION. <u>$H(\sigma)$, half-twist defined by σ</u>

Let D, K be as above. Let $a, b \in K$, and let σ be a smooth simple path in $\text{Int}(D)$ connecting a with b s.t. $\sigma \cap K = \{a, b\}$. Choose a small regular neighborhood U of σ contained in $\text{Int}(D)$, s.t. $U \cap K = \{a, b\}$ and an orientation preserving diffeomorphism $f : \mathbb{R}^2 \longrightarrow \mathbb{C}^1$ ($\mathbb{C}^1$ is taken with usual "complex" orientation) such that $f(\sigma) = [-1, 1]$, and $f(U) = \{z \in \mathbb{C}^1 \mid |z| < 2\}$. Let $\alpha(r), r \geqslant 0$, be a real smooth monotone function such that $\alpha(r) = 1$ for $r \in [0, \frac{3}{2}]$ and $\alpha(r) = 0$ for $r \geqslant 2$.

Define a diffeomorphism $h : \mathbb{C}^1 \longrightarrow \mathbb{C}^1$ as follows. For $z \in \mathbb{C}^1$, $z = re^{i\varphi}$, let $h(z) = re^{i(\varphi+\alpha(r)\pi)}$. It is clear that on $\{z \in \mathbb{C}^1 \mid |z| \leq \frac{3}{2}\}$, $h(z)$ is the positive rotation by 180° and that $h(z) = \text{Identity}$ on $\{z \in \mathbb{C}^1 \mid |z| \geq 2\}$, in particular, on $\mathbb{C}^1 \setminus f(U)$. Considering $(f \circ h \circ f^{-1})|_D$ (we always take composition from left to right), we get a diffeomorphism of D which switches a and b and is the identity on $D \setminus U$. Thus it defines an element of $B_n[D, K]$, called *the half-twist defined by σ and denoted $H(\sigma)$.*

The following is the basic braid monodromy associated to a curve with single singularity.

PROPOSITION - EXAMPLE 2.1. *Let $E = \{x \in \mathbb{C} \mid |x| \leq 1\}$, $D = \{y \in \mathbb{C} \mid |y| \leq R\}$, $R > 1$, S is the curve $y^2 = x^\nu$, $u = 1$. Clearly, here $n = 2, N = \{0\}$, $K = \{-1, +1\}$ and $\pi_1(E \setminus N, 1)$ is generated by $\Gamma = \partial E$ (positive orientation). Denote by $\varphi : \pi_1(E \setminus N, 1) \to B_2[D, K]$ the braid monodromy of S w.r.t. $E \times D, \pi, u$. Then $\varphi(\Gamma) = H^\nu$, where H is the positive half-twist defined by $[-1, 1]$ ("positive generator" of $B_2[D, K]$).*

PROOF. We can write $\Gamma = \{e^{2\pi it}, t \in [0, 1]\}$. Lifting Γ to S we get two paths:

$$\delta_1(t) = \left(e^{2\pi it},\ e^{2\pi i\nu t/2}\right)$$

$$\delta_2(t) = \left(e^{2\pi it},\ -e^{2\pi i\nu t/2}\right).$$

Projecting $\delta_1(t)$, $\delta_2(t)$ to D we get two paths:

$$a_1(t) = e^{\pi it\cdot\nu}, \qquad 0 \le t \le 1$$
$$a_2(t) = -e^{\pi it\cdot\nu}, \qquad 0 \le t \le 1.$$

This pair of paths (a_1, a_2), each composed of ν consecutive half-circles, defines a motion of $\{1, -1\}$ in D. This motion is the ν-th power of the motion defined by:

$$b_1(t) = e^{\pi it}, \qquad 0 \le t \le 1$$
$$b_2(t) = -e^{\pi it}, \qquad 0 \le t \le 1.$$

The braid of $B_2[D, \{1, -1\}]$ induced by this last motion, coincides with the half-twist H corresponding to $[-1, 1] \subset D$. Thus $\varphi(\Gamma) = H^\nu$. □

Proposition-Definition 2.2. (Dehn-twist) *Denote by d the element of $\pi_1(D \setminus K, u)$ represented by the loop ∂D (with positive orientation). There exists a unique element of B_n, denoted by Δ_n^2 or $\Delta_n^2[D, K]$, such that for any Γ, a simple loop around a single point of K, the (right) action of Δ_n^2 (as an element of B_n) on Γ is as follows:*

$$\Gamma \cdot \Delta_n^2 = d\Gamma d^{-1}\,.$$

Δ_n^2 is called a Dehn-twist.

Proof. [**MoTe3**], V.2.1.

Remark. Clearly, Δ_n^2 acts as a full-twist around all the points of K. One can justify the notation Δ_n^2, but here we prefer to simply use it as a notation. More about Δ_n^2 can be found in [**MoTe3**] and [**Te6**].

Proposition 2.3. (a) $\Delta_n^2 \in \mathrm{Center}(B_n)$. (b) *$\Delta_n^2$ is a product of $n(n-1)$ half-twists.*

Proof. [**MoTe3**], V.4.1 and V.2.2.

Proposition - Example 2.4. *Let S be a union of p lines, meeting in one point $s_0, s_0 = (x_0, y_0)$. Let $D, E, u, K = K(u)$ be as before. Let φ be the braid monodromy of S w.r.t. $E \times D, \pi, u$. Clearly, here $N = \{x_0\}$ (a single point) and $\pi_1(E \setminus N, u)$ is generated by $\Gamma = \partial E$. Then $\varphi(\Gamma) = \Delta_p^2 = \Delta_p^2[D, K(u)]$.*

Proof. By a continuous change of s_0 and of the n lines passing through s_0 (and by uniqueness of Δ_p^2) we can reduce the proof to the following case: $S = \bigcup L_k$, $L_k\colon\ y = j_k x,\ j_k = e^{2\pi ik/p},\ k = 0, \ldots, p-1$. Then $N = \{0\}$. We can take $u = 1$, $\Gamma = \{x = e^{2\pi it},\ t \in [0, 1]\}$, $K = \{j_k \mid k = 0 \ldots p-1\}$. Lifting ∂E to S and then projecting it to D, we get n loops:

$$a_k(t) = e^{2\pi i(t + k/p)}, \quad k = 0, \ldots, p-1, \quad t \in [0, 1].$$

Thus the motion of the points $a_k(0) = j_k$, represented by the corresponding loops $a_k(t)$ (for $k = 0, \ldots, p-1$), is a full-twist which defines the braid $\Delta_p^2[D, \{a_k(0)\}] = \Delta_p^2[D, K(1)]$. (To check the last fact, see the corresponding actions in $\pi_1(D \setminus K, u)$). □

Definition. Braid monodromy of a projective curve

Let B be an algebraic curve of degree p in $\mathbb{CP}^2$. Choose generically a line L at infinity ($\#(L \cap B) = p$) and affine coordinates (x, y) in $\mathbb{C}^2 = \mathbb{CP}^2 \setminus L$ so that the coordinate projection, $\pi : \mathbb{C}^2 \to \mathbb{C}$, $\pi(x, y) = x$, induces a generic map from $B \cap \mathbb{C}^2$ to $\mathbb{C}$ by restriction (in particular, the center of this projection in $\mathbb{CP}^2$ must lie outside of B). Let $N = \{x \in \mathbb{C} \mid \pi^{-1}(x) \cap B \lneqq p\}$, E be a closed disk on the x-axis with $N \subset \operatorname{Int}(E), D$ be a sufficiently large closed disk on the y-axis s.t. $\pi^{-1}(E) \cap B \subset E \times D$. Choose $u \in \partial E$. Denote by $S = B \cap (E \times D)$. *The braid monodromy of B w.r.t. L, u* is the braid monodromy of S w.r.t. $E \times D, \pi, u$, i.e., the homomorphism

$$\varphi : \pi_1(E \setminus N, u) \to B_p[D, K]$$

We recall the notion of a geometric free base of the fundamental group of a punctured disk in $\mathbb{C}$ and a basic property of it. Since we shall choose such bases both for the x-axis and the y-axis, we make independent notations.

Definition. A bush

Let U be a closed disk in $\mathbb{C}$ and F a finite set in $Int(U)$, $F = \{w_1, \ldots, w_n\}$, $v \in \partial U$.

Consider in U an ordered set of simple paths $(T_1, \ldots, T_n)$ connecting the points $w_1, \ldots, w_n$ with v such that

1. $T_i \cap T_j = \{v\}$ if $i \neq j$;
2. Each path T_i intersects a small circle around v in a single point u'_i, and the order of these points on the circle is given by the positive ("counterclockwise") orientation.

We say that two such sets $(T_1, \ldots, T_n)$ and $(T'_1, \ldots, T'_n)$, are equivalent if on the homotopy class level we have

$$\ell(T_i) = \ell(T'_i) \qquad (\text{for } i = 1, \ldots, n)$$

where $\ell(T_i)$ is a closed loop based at v, then following the path T, then encircles w_i counterclockwise and returns (see Fig. 2.2). An equivalence class of such sets is called *a bush* in $(U \setminus F, v)$. The bush represented by $(T_1, \ldots, T_n)$ is denoted by $\langle T_1, \ldots, T_n \rangle$.

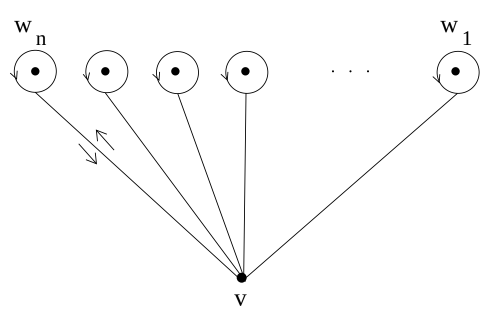


Fig. 2.2

Definition. <u>geometric base, g − base</u>

Let U, F, v, be as above. A *g-base of* $\pi_1(U \setminus F, v)$ is an ordered free base of $\pi_1(U \setminus F, v)$ which has the form $(\ell(T_1), \ldots, \ell(T_n))$ where $\langle T_1, \ldots, T_n \rangle$ is a bush in $U \setminus F$ (see fig. 2.2).

Proposition 2.5. *Let B be an algebraic curve of degree p in $\mathbb{CP}^2$. Let $L, \pi, u, D, E, K(u)$ be as in the beginning of §2. Let φ be the braid monodromy of B w.r.t. L, π, u. Let $(\delta_1, \ldots, \delta_r)$ be a g-base of $\pi_1(E \setminus N, u)$ $(r = \#N)$. Then*

$$\prod_{i=1}^{r} \varphi(\delta_i) = \Delta_p^2 = \Delta_p^2[u \times D, K(u) \cap B].$$

Proof. One can see that $\prod\limits_{i=1}^{r} \delta_i = \partial E$ is positively oriented (see also [**MoTe3**], §2). Thus we have to prove that $\varphi(\partial E) = \Delta_p^2$. We can assume E arbitrarily big, so that ∂E will be very close to ∞ at the x-axis. Continuously deforming coefficients in the equations of B such that new resulting curves always remain transversal to L, we can reduce the proof to the case where B is a union of n lines intersecting at a single point. Now use Proposition - Example 2.4. □

Following Proposition 2.5 we define:

Definition. <u>Braid monodromy factorization of Δ_p^2</u> (associated to a plane projective curve)

Braid monodromy factorization of Δ_p^2 (associated to a plane projective curve) is a product of the form $\Delta_p^2 = \prod\limits_i \varphi(\delta_i)$, where φ is the braid monodromy of the projective curve and $\{\delta_i\}$ is a g-base of $\pi_1(E \setminus N, u)$.

Remarks.

(1) A braid monodromy factorization depends, in fact, not only on the curve but also on the choice of the base. When needed, we then refer to *braid monodromy factorization of Δ_p^2 associated to a curve and a base $\{\delta_i\}$.*

(2) In the other direction, a g-base of $\pi_1(E \setminus N, u)$ and the corresponding factorization determine the braid monodromy. (The values of a homomorphism on a base determine the homomorphism.) For applications, it is usually sufficient to know a a factorization, without referencing to a particular g-base (like in the proof of 3.2 below or in [MoTe7] or in [Te5]).

(3) For a nonsingular B, each $\varphi(\delta_i)$ is a (positive) half-twist in B_p. (See [**MoTe3**], Prop. IV.1.1). The associated factorization is then called *prime*.

(4) A braid monodromy factorization is a presentation of Δ_p^2 as a product of (positive) elements in B_p^+. Not all factorizations of Δ_p^2 to products of positive elements are induced from a curve. (The definition of positive elements and the semigroup B_p^+ can be found in [**MoTe3**], §5; roughly, these are products of positive half-twists).

Proposition 2.6. *Let B be a cuspidal curve in $\mathbb{CP}^2$ (that is, all singularities of B are locally of the form $y^2 = x^2$ (a node) or $y^2 = x^3$ (a cusp)). Then any braid monodromy factorization of Δ_p^2 (associated to B), can be written as a product $\Delta_p^2 = \prod\limits_i (Q_i^{-1} H_1^{\nu_i} Q_i)$ of suitable conjugates of some fixed positive half-twist H_1, raised to some power $\nu_i = 1, 2,$ or 3.*

Proof. Recall that we are using generic projections of $\mathbb{C}^2 \xrightarrow{\pi} \mathbb{C}$ w.r.t. the projective curve. Each singularity of $\pi|_B$ is of the type $y^2 = x^\nu$, $\nu = 1, 2$, or 3. Now use Proposition - Example 2.1 to get $\varphi(\delta_i) = H_i^{\nu_i}$, with $\nu_i = 1, 2$, or 3, and where H_i is a half-twist. Every two half-twists in B_p are conjugate, so for every i, there exists a Q_i s.t. $H_i = Q_i^{-1} H_1 Q_i$. Thus, $\Delta_p^2 = \Pi\varphi(\delta_i) = \Pi(Q_i^{-1} H_1^{\nu_i} Q_i)$. □

Remark. We can take any half-twist for H_1.

In the next section we shall consider a braid monodromy factorization related to Hirzebruch surfaces.

§3. Braid Monodromy Related to a Generic Projection of $F_{k(a,b)}$

Let $S_{k(a,b)}$ be the branch curve of a generic projection of $F_{k(a,b)}$ to $\mathbb{CP}^2$. We want to compute the braid monodromy of $S_{k(a,b)}$. We believe that the "braid monodromy type" of a branch curve determines the "deformation type" of the related surface. Thus our main goal in computing the braid monodromy of a branch curve is to distinguish between surfaces which are not a deformation of each other (see [**Te3**]). Since $F_{k(a,b)}$ can be deformed to $F_{k-2(a',b')}$ (see [**FRoTe**]), it is enough to consider the case $k = 0$ and $k = 1$. The case $k = 0$ was presented in [**MoTe1**]; the case $k = 1$ will be described here. Theorem 3.2 gives a braid monodromy factorization for $S_{1(a,b)}$, and thus determine the braid monodromy type of $S_{1(a,b)}$. The nonspecialist might want to skip the details of this theorem, and the subsequent explanation while realizing that we heavily use the degeneration from §1 in the calculation.

Before we state Theorem 3.2, we describe in greater detail the branch curve of the degenerated object. We shall use the degeneration of $F_{1(a,b)}$ described in §1. Recall that $F_{1(a,b)}$ is degenerated to $F^0_{1(a,b)}$, a union of planes in the following configuration (here $b = 5$, $a = 4$):

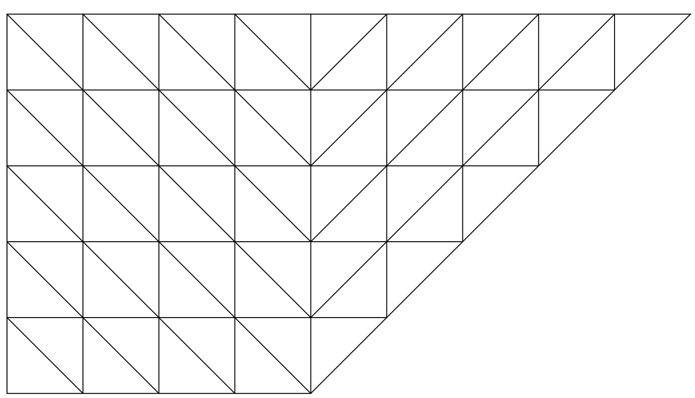


Fig. 3.1

Each triangle represents a plane and each inner edge represents an intersection line between planes. The number of planes is $2ab+b^2$ and the number of intersection lines is $3ab - a + \frac{3b}{2}(b-1)$. We take a generic projection of $F^0_{1(a,b)}$ onto $\mathbb{CP}^2$ where each plane is projected onto $\mathbb{CP}^2$. The ramification curve of this projection is the union of lines. The singular points of the ramification curve are represented by

vertices. The branch curve of $F^0_{1(a,b)} \to \mathbb{CP}^2$, denoted $S^0_{1(a,b)}$, is the image of the union of lines and its singular points are the images of the vertices and the intersection points in $\mathbb{CP}^2$ of the images of any two of the intersection lines.

We numerate the vertices $a_1, \dots, a_{\nu_0}$ from right to left, from bottom to top (including two points a_1 and $a_{\nu_0 - b}$ which are not on S_0 and two points a_{m_0+b} and a_{ν_0} which are on S_0 but not singular points of S_0), where $m_0 = \frac{b(b+1)}{2} + 1$ and $\nu_0 = m_0 + a(b+1) + b$ $(= \frac{b(b+1)}{2} + (a+1)(b+1))$. We numerate the lines in lexicographic order from the bigger index to the smaller one: $L_1, \dots, L_{p_0}$, where $p_0 = \frac{1}{2}(6ab - 2a - 3b + 3b^2)$. See Fig. 3.2.

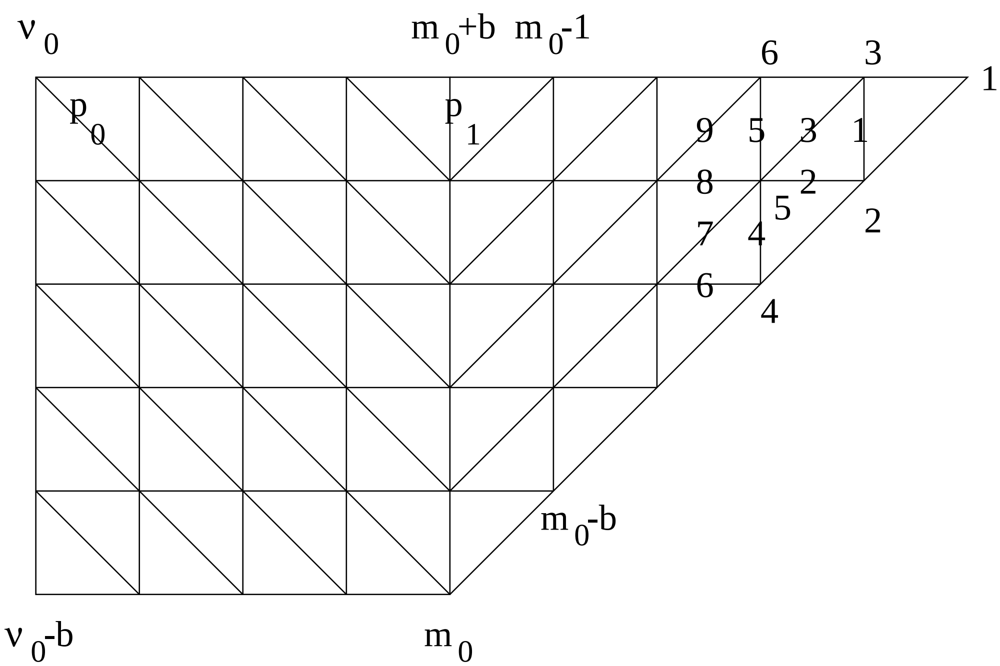


Fig. 3.2

Lemma 3.1. $p = \deg S_{1(a,b)} = 6ab - 2a - 3b + 3b^2$.

Proof. Lemma 7.1.3(b) in [**MoRoTe**] for $k = 1$.

Remark. In [**MoTe4**], §2,§3 (see also [**MoTe6**], §1), we introduced a *regeneration process* for "reconstructing" branch curves from the branch curve of the degenerated object. Since lines are doubled during the regeneration process, $\deg S_{1(a,b)} = 2 \deg S^0_{1(a,b)}$. So one can also get the Lemma by doubling the number of intersection lines in the above configuration.

Let ℓ_j and b_i be the images of L_j and a_i in $\mathbb{CP}^2$ and $q_i = \ell_i \cap \mathbb{C}^1_u$ (see §2).

Theorem 3.2. *The braid monodromy factorization of Δ^2_p associated to $S_{1(a,b)}$ (where $p = \deg S_{1(a,b)}$) is as follows: $\Delta^2_p = \prod\limits_{i=1}^{\nu_0} \tilde{C}_i \tilde{P}_i$; $\tilde{P}_i$ is the local braid monodromy factorization around b_i; and $\tilde{C}_i = \prod \tilde{Z}^2_{ii',jj'}$ for $i < j$, $L_i \cap L_j = \emptyset$, and*

$\tilde{Z}_{i,j}$ is a half-twist from q_i to q_j corresponding to the path described in Fig. 3.3*, j_0 is the smallest index s.t. L_{j_0} meets L_j in the vertex with the higer index, and $\tilde{Z}^2_{ii',jj'} = \tilde{Z}^2_{ij}\tilde{Z}^2_{ij'}\tilde{Z}^2_{i'j}\tilde{Z}^2_{i'j'}$ (a product of 4 full-twists).*

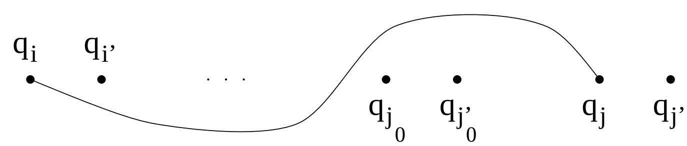


FIG. 3.3

PROOF. We use the notation of §2. Recall that $F_{1(a,b)}$ was degenerated to $F^0_{1(a,b)}$ with the branch curve $S_{1(a,b)}$ of the generic projection onto $\mathbb{CP}^2$ being degenerated to $S^0_{1(a,b)}$, the branch curve of $F^0_{1(a,b)} \to \mathbb{CP}^2$. As explained earlier, $S^0_{1(a,b)}$ is an arrangement of p_0 lines. In fact, $p_0 = p/2$ (see the above remark). In this arrangement, no 3 vertices of higher multiplicity (where 3 lines or more meet) are collinear. In [**MoTe3**], §9, we computed the braid monodromy factorization associated to such line arrangements. It can be presented as:

$$\Delta^2_{p_0} = \prod_{i=1}^{\nu_0} C_i \Delta^2_{k_i}[\mathbb{C}^1_{u_0}, S_i \cap C^1_{u_0}]$$

$$C_i = \prod_{i<j,\ L_i\cap L_j=\emptyset} \tilde{Z}^2_{ij}$$

where $S_i = \{\text{lines through } a_i\}$, with $k_i = \#S_i$, and $\Delta^2_{k_i}$ = Dehn-twist around $S_i \cap \mathbb{C}^1_{u_0}$. (The notation $\tilde{Z}_{ij}$ is explained in the formulation of the Theorem.)

By Proposition-Example 2.4, $\Delta^2_{k_i}$ is, in fact, the local braid monodromy of $S^0_{1(a,b)}$ around b_i ($\Delta^2_i = 1$ for $i = 1, m_0 + b, \nu_0 - b$ and ν_0).

We apply the regeneration process (from [**MoTe4**]) on $\prod\limits_{i=1}^{\nu_0} C_i\Delta^2_i$ to get $\prod\limits_{i=1}^{\nu_0} \tilde{C}_i\tilde{P}_i$. (See [**MoTe4**], §2 for the starting situation and §3 for the regeneration rules I, II, and III (Lemmas 3.1, 3.2 and 3.3)). When "regenerating," lines are "doubled" and each point q_j in the typical fiber is replaced by two points q_j and $q_{j'}$. The product $\tilde{C}_i$ is easy to describe. It is the result of applying the second regenerating rule on C_i, i.e., each full-twist $\tilde{Z}^2_{ij}$ is replaced by the product of the 4 full-twists $\tilde{Z}^2_{ij}, \tilde{Z}^2_{ij'}, \tilde{Z}^2_{i'j}$, $\tilde{Z}^2_{i'j'}$ in this order (written in short $Z^2_{ii',jj'}$) (see also C-table in [**MoTe1**]).

For a_i's which are not singular points of $S^0_{1(a,b)}$, we get the following expression: $\tilde{P}_{m_0+b} = Z_{p_1p'_1}$ $\left(p_1 = \frac{b(3b-1)}{2}\right)$, $\tilde{P}_{\nu_0} = Z_{p_0p'_0}$ (by Proposition 5.2.2 of [**MoRoTe**]), and $\tilde{P}_1 = \tilde{P}_{\nu_0-b} = 1$.

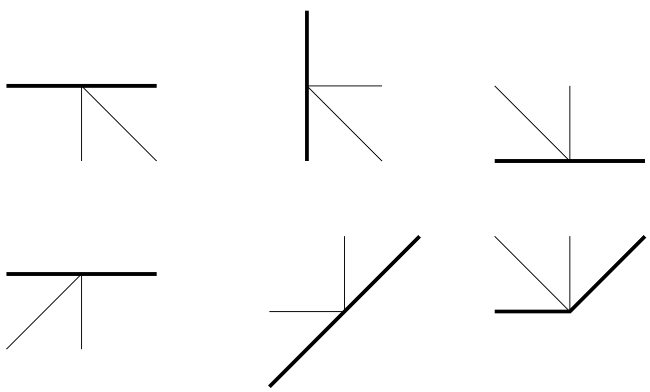
One can compute the order of each $\tilde{P}_i$ (in terms of the number of positive half-twists that appear in the presentation) which is 132 ($= 12 \cdot 11$) for a 6-point, and 12 ($= 3 \cdot 4$) for a 3-point. (Since this proof is for the specialist, I shall not give details of the calculations). We sum up the degree of all factors in $\prod\limits_{i=1}^{\nu_0} \tilde{C}_i\tilde{P}_i$, and get $p(p-1)$. A priori, $\prod\limits_{i=1}^{\nu_0} \tilde{C}_i\tilde{P}_i$ is part of a braid monodromy factorization of Δ^2_p,

associated to $S_{1(a,b)}$. Since the degree of Δ_p^2 is exactly $p(p-1)$ (Proposition 2.3), we get $\Delta_p^2 = \prod\limits_{i=1}^{\nu_0} \tilde{C}_i \tilde{P}_i$, and thus there are no extra factors in the braid monodromy factorization; and $\tilde{P}_i$ is the local braid monodromy around b_i. □

Remark. The sources of $\tilde{C}_i$ are the intersection of the lines ℓ_i and ℓ_j, for i and j, such that L_i and L_j do not intersect.

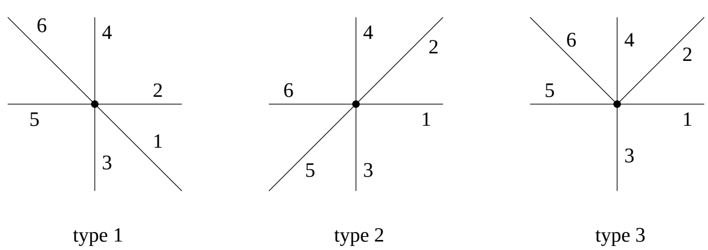
About the computation of $\tilde{P}_i$ from Theorem 3.2

Each singular point of S_0 is either a 3-point (lies on 3 planes and 2 lines) or a 6-point (lies on 6 planes and 6 lines). Two intersection lines meet in each 3-point, and 6 intersection lines meet in the 6-point. Different types of 3-points, 6-points are presented in Fig. 3.4.



3-points

Fig. 3.4(a)



6-points

Fig. 3.4(b)

(The intersection lines are thinner; the thick line is the border of the configuration.) (*Warning*: In [**MoTe1**] and [**MoTe3**], we refer to 3-points as 2-points, i.e., by the number of lines and not by the number of planes.)

The difference between the various 3-points lies in the order in which the lines appear in the degeneration process. More precisely, whether the smaller indexed

line is a diagonal, vertical, or a horizontal line and whether the 2 lines meet in the endpoint with higher index of both, or in the endpoint with smaller index of both. This difference affects the local braid monodromy around each point.

By [**MoRoTe**], Prop. 4.4.1 for a_i a 3-point, $a_i = L_j \cap L_k$ (L_k diagonal) we have $\tilde{P}_i = Z^{(3)}_{k,jj'} \cdot \tilde{Z}_{kk'}$ where:

$$\tilde{Z}_{kk'} = \begin{cases} (Z_{kk'})_{Z_{k'j} Z^{-1}_{jj'} Z_{j'k}} & k < j \\ (Z_{kk'})_{Z_{kj} Z^{-1}_{jj'} Z_{jk'}} & k > j \end{cases}$$

where Z_{ij} is the half-twist corresponding to a path which connects q_i with q_j from below the real line (q_i are real), and $(A)_B = B^{-1}AB$ is a conjugation symbol. (Note that a half-twist conjugated by a half-twist gives a third half-twist).

$$Z^{(3)}_{k,jj'} = Z_{kj} \cdot Z_{kj'} \cdot (Z_{kj'})_{Z_{jj'}}.$$

is a short notation for product of three half-twists.

Concerning 6-points, we have three types; each one has six lines meeting in one point. We introduce a local numeration on each configuration which is compatible with the global ordering.

In [**MoTe1,MoTe6**] a complete computation for $\tilde{P}_i$ where a_i is a 6-point of type 1 is given (table Δ^2_α, α 6-point in [**MoTe1**] and Lemma 1.1 in [**MoTe6**]). In [**MoTe1**] the local numeration is described by Fig. 3.5:

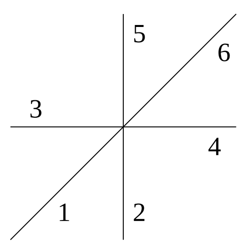


FIG. 3.5

which is obtained by a 90° clockwise turn of the diagram describing a 6-point of type 1. Thus $\tilde{P}_i$ for this type is determined by the computations given there.

A 6-point of type 2 or 3 is different; no exchange of numeration will result in the "classical" 6-point from [**MoTe1**] with the same order of regeneration. For these cases, the computations will appear in [**AmTe**] and [**CiMiTe**].

§4. Fundamental Groups of Galois Covers of Hirzebruch Surfaces

After computing the braid monodromy of a curve S in $\mathbb{CP}^2$, we can use the Zariski-Van Kampen theorem ([**VK**], (cf., for example, [**Te1**])) to get a finite presentation for the fundamental group of its complement. If S is a branch curve of a generic projection from a surface, the fundamental group $\pi_1(\mathbb{CP}^2 \setminus S)$ can yield a new invariant of a surface (see [**Te3**]), using the fact that in many cases such groups are almost polycyclic. Moreover, the fundamental group of the Galois cover of the surface is isomorphic to a quotient of a subgroup of $\pi_1(\mathbb{CP}^2 \setminus S)$, so we can recover such fundamental groups of surfaces and in particular those of the Galois cover of Hirzebruch surfaces.

Let us recall the definition of a Galois cover:

DEFINITION. Galois cover w.r.t. to generic projection

Let X be a surface and let $f : X \to \mathbb{CP}^2$ be a generic projection of $\deg n$. Let $\underbrace{X \underset{f}{\times} \cdots \underset{f}{\times} X}_{n}$ be the fibered product,

$$X \underset{f}{\times} \ldots \underset{f}{\times} X = \{(x_1, \ldots, x_n) \bigm| \forall i, j, \quad f(x_i) = f(x_j)\};$$

and let Δ be the "big" diagonal,

$$\Delta = \{(x_1, \ldots, x_n) \bigm| \exists\, i, j \quad x_i = x_j\}.$$

Then we define the Galois cover X_{Gal} of X to be the surface

$$X_{\mathrm{Gal}} = \overline{X \underset{f}{\times} \ldots \underset{f}{\times} X \setminus \Delta}.$$

There is a natural projection $\tilde{f} : X_{\mathrm{Gal}} \to \mathbb{CP}^2$ (projection on the first coordinate).

The following theorem is concerned with the Galois cover of Hirzebruch surfaces $F_{k(a,b)}$ (see §1).

THEOREM 4.1. $\pi_1(F_{k(a,b)})_{\mathrm{Gal}} = (\mathbb{Z}_c)^{n-2}$, *where* $c = \gcd(a, b)$, $n = \deg F_{k(a,b)} = 2ab + kb^2$.

PROOF. See [**MoRoTe**] and [**FRoTe**]. Here we shall only recall the connection of $\pi_1(X_{\mathrm{Gal}})$ with $\pi_1(\mathbb{CP}^2 \setminus S)$, for S the branch curve of $X \xrightarrow{f} \mathbb{CP}^2$ generic. Let us generically choose an affine piece $\mathbb{C}^2$ of $\mathbb{CP}^2$. Let $X_{\mathrm{Gal}}^{\mathrm{Aff}}$ be the part of X_{Gal} lying over it.

There is a natural epimorphism $\pi_1(\mathbb{C}^2 \setminus S, u_0) \xrightarrow{\psi} S_n$ for u_0 any point not in S and S_n the symmetric group on $n = \deg f$ objects. In fact, lifting a loop at u_0 to n paths in X, induces a permutation of $f^{-1}(u_0)$. Since $\# f^{-1}(u_0) = n$, we thus get an element of S_n. Clearly, ψ is surjective. So we have an exact sequence $1 \to \ker\psi \to \pi_1(\mathbb{C}^2 \setminus S, u_0) \to S_n \to^{-1}$ of groups.

In order to establish an isomorphism of $\pi_1(X_{\mathrm{Gal}}^{\mathrm{Aff}})$ with a quotient of a subgroup of $\pi_1(\mathbb{CP}^2 \setminus S)$, we have to choose a certain system of generators for $\pi_1(\mathbb{C}^2 \setminus S, u_0)$. Let $p, u, K, \mathbb{C}_u^1$ be as in §2.

Let $\{\Gamma_j\}_{j=1}^p$ be a g-base of $\pi_1(\mathbb{C}_u^1 \setminus K)$. (Recall from §2 that $\{\Gamma_j\}$ is a free base). There is a natural surjection $\pi_1(\mathbb{C}_u^1 \setminus S, u) \xrightarrow{\mu} \pi_1(\mathbb{C}^2 \setminus S, u)$ induced from the inclusion $\mathbb{C}_u^1 \setminus S \hookrightarrow \mathbb{C}^2 \setminus S$. By abuse of notation, we shall denote the image of Γ_j in $\pi_1(\mathbb{C}^2 \setminus S)$ also by Γ_j. Clearly, the set $\{\Gamma_j\}_{j=1}^p$ then generates $\pi_1(\mathbb{C}^2 \setminus S, u)$.

Since f is stable, the ramification is of order 2 and $\psi(\Gamma_j)$ is a transposition in S_n. So $\Gamma_j^2 \in \ker\psi$. Let $\mathbf{\Gamma}$ be the normal subgroup generated by $\{\Gamma_j^2\}_{j=1}^p$. Then $\mathbf{\Gamma} \subseteq \ker\psi$. By the standard isomorphism theorems, we have:

$$1 \to \frac{\ker\psi}{\mathbf{\Gamma}} \to \frac{\pi_1(\mathbb{C}^2 \setminus S, u)}{\mathbf{\Gamma}} \to S_n \to 1.$$

In [**MoTe1**],0.3, we proved $\pi_1(X^{\mathrm{Aff}}_{\mathrm{Gal}}) \simeq \frac{\ker\psi}{\mathbf{\Gamma}}$. In [**MoTe8**], we considered the projective case and proved that

$$\pi_1(X_{\mathrm{Gal}}) \simeq \frac{\ker\psi}{\left\langle \mathbf{\Gamma}, \prod_{j=1}^{q} \Gamma_j \right\rangle}$$

This established the connection between $\pi_1(\mathbb{C}^2 \setminus S)$ and $\pi_1(X_{\mathrm{Gal}})$. The actual deduction of $\pi_1(X_{\mathrm{Gal}})$ from $\pi_1(\mathbb{C}^2 \setminus S)$ involves the Reidemeister-Schreier method from [**KMS**].

□

Corollary 4.2. *$(F_{k(a,b)})_{\mathrm{Gal}}$ is simply connected iff a, b are relatively prime.*

§5. Chern Numbers of Galois Covers of Hirzebruch Surfaces

For any generic (stable, finite) morphism $g : X \to \mathbb{CP}^2$, from a nonsingular algebraic surface, it can be shown that the induced X_{Gal} is nonsingular (see [**Te5**]). Moreover, if $S \subset \mathbb{CP}^2$ is the branch curve of g, and $\tilde{S} \subset X_{\mathrm{Gal}}$ is the ramification curve of $\tilde{g} : X_{\mathrm{Gal}} \to \mathbb{CP}^2$ and ℓ a line on $\mathbb{CP}^2$, then the canonical class $K_{X_{\mathrm{Gal}}}$ of X_{Gal} is equal to $g^*(-3\ell) + \tilde{S}$. On the level of divisor classes, we have $\tilde{S} = \frac{1}{2}g^*(S) = \frac{1}{2}mg^*(\ell)$, with $m = \deg S$. Hence for the canonical class, we get $K_{X_{\mathrm{Gal}}} = \left(\frac{m}{2} - 3\right) g^*\ell$. Thus when $m > 6$, the bundle $K_{X_{\mathrm{Gal}}}$ is ample and X_{Gal} is a minimal surface of general type. Moreover, X_{Gal} is a spin manifold iff $K_{X_{\mathrm{Gal}}}$ is even iff m is not a multiple of 4.

Notation. Let us denote for short $Y_{k(a,b)} = (F_{k(a,b)})_{\mathrm{Gal}}$.

By the above and Lemma 3.1 we get

Corollary 5.1. *$Y_{k(a,b)}$ is of general type if and only if one of the following is true:*
(i) $k = 0$; $ab \geq 3$;
(ii) $k = 1, 2$; $ab \geq 2$;
(iii) $k \geq 3$.
$Y_{k(a,b)}$ is a spin manifold when any of the following is true:
(i) $b \equiv 0(4)$, $a \equiv 1(2)$;
(ii) $b \equiv 1(4)$, $k \equiv 0(2)$;
(iii) $b \equiv 2(4)$, $a + k \equiv 1(2)$;
(iv) $b \equiv 3(4)$.

Remarks. (1) There are other $Y_{k(a,b)}$ which admit a spin structure.
(2) Corollary 5.1 is in fact a restatement of Theorem 0.3 b) and c) from [**MoRoTe**].

We shall give here a formula from [**Te2**] for the Chern numbers of X_{Gal} in terms of certain invariants of X.

Theorem 5.2. *Let E be the hyperplane section and K the canonical divisor of X, and let $n = \deg X$. Then*

$$c_1^2(X_{\mathrm{Gal}}) = \frac{n!}{4}[(E \cdot K)^2 + 6n(E \cdot K) + 9n^2 - 12(E \cdot K) - 36n + 36],$$

$c_2(X_{\mathrm{Gal}}) =$
$$\frac{n!}{24}[72 - 10c_1^2(X) - 54(E\cdot K) - 114n + 27n^2 + 14c_2(X) + 3(E\cdot K)^2 + 18n(E\cdot K)].$$

Proof. [**Te2**], Proposition 2.1.

Proposition 5.3. *For $a, b \geq 1$ and $n = 2ab + kb^2 (= \deg F_{k(a,b)})$, we have:*

$$c_1^2(Y_{k(a,b)}) = \frac{n!}{4}\left\{\begin{array}{l} 4a^2 + 4b^2 - 64ab + 24a + 24b - 24a^2b - 24ab^2 \\ +\, 36 + 36a^2b^2 + k(12b + 4ab - 12b^3 + 36ab^3 - 24ab^2 \\ -\, 32b^2) + k^2(b^2 - 6b^3 + 9b^4) \end{array}\right\}$$
$$= \frac{n!}{4}\left\{k^2b^2(3b-1)^2 + 4kb(3b-1)(3ab - a - b - 3) + 4(3ab - a - b - 3)^2\right\},$$

$$c_2(Y_{k(a,b)}) = \frac{n!}{8}\left\{\begin{array}{l} 4(4 + 9a + 9b - 17ab + a^2 + b^2 + 9a^2b^2 - 6a^2b - 6ab^2) \\ +\, 2k(9b - 17b^2 + 18ab^3 + 2ab - 12ab^2 - 6b^3) \\ +\, k^2(9b^4 + b^2 - 6b^3) \end{array}\right\}$$
$$= \frac{n!}{8}\left\{(3b-1)^2(2a + kb)^2 + (9 - 17b - 6b^2)(4a + 2kb) + 4(b^2 + 9b + 4)\right\}.$$

Proof. We have here:

$$\begin{aligned} &E(F_{k(a,b)}) = aC + bE_0, \\ &K(F_{k(a,b)}) = -2E_0 + (k-2)C, \\ &c_1^2(F_{k(a,b)}) = 8, \\ &c_2(F_{k(a,b)}) = 4, \\ &C\cdot E_0 = 1, \quad E_0^2 = k, \quad C^2 = 0. \end{aligned}$$

Thus,
$$E\cdot K = -2a - 2b - bk,$$
$$n = \deg F_{k(a,b)} = E^2 = 2ab + b^2k.$$
We substitute this in the formulas from Theorem 5.2 to get the proposition. □

Corollary 5.4. *For $k = 1$, $a \geq 1$, $b \geq 1$ we have:*

$$c_1^2(Y_{1(a,b)}) = \frac{(2ab + b^2)!}{4}(3b^2 + 6ab - 3b - 2a - 6)^2,$$
$$c_2(Y_{1(a,b)}) = \frac{(2ab + b^2)!}{8}\left\{\begin{array}{l} 16 + 54b + 36a - 64ab + 4a^2 - 29b^2 + 36a^2b^2 \\ -\, 24a^2b - 48ab^2 - 18b^3 + 9b^4 + 36ab^3 \end{array}\right\}.$$

For $k = 0$, $a \geq 1$, $b \geq 1$ we have:

$$c_1^2(Y_{0(a,b)}) = (2ab)!(3ab - a - b - 3)^2,$$
$$c_2(Y_{0(a,b)}) = \frac{(2ab)!}{2}\{4 + 9a + 9b - 17ab + a^2 + b^2 + 9a^2b^2 - 6a^2b - 6ab^2\}.$$

Using Theorem 4.1 and Corrolary 5.4, one can get examples of surfaces with the same Chern numbers and different fundamental groups:

THEOREM 5.5. *Let s, t be odd relatively prime positive numbers, then*

$$\pi_1(Y_{1(s,2t)}) = 0, \qquad \pi_1(Y_{0(s+t,2t)}) = (\mathbb{Z}_2)^{4st+4t^2-2},$$

$$c_1^2(Y_{1(s,2t)}) = c_1^2(Y_{0(s+t,2t)}) = (4st+4t^2)! \left\{ \begin{array}{l} 9 + 6s + 18t + s^2 - 30st \\ -27t^2 - 12ts^2 - 48t^2s - 30t^3 \\ +36t^2s^2 + 72st^3 + 36t^4 \end{array} \right\},$$

$$c_2(Y_{1(s,2t)}) = c_2(Y_{0(s+t,2t)}) = \frac{(4st+4t^2)!}{2} \left\{ \begin{array}{l} 4 + 27t + 9s - 32st \\ + s^2 - 29t^2 + 36s^2t^2 - 12s^2t \\ -48st^2 - 36t^3 + 36t^4 + 72st^3 \end{array} \right\}.$$

Using the Hirzebruch formulae for the signature of a surface in terms of the Chern numbers, $\tau(Y) = \frac{1}{3}(c_1^2(Y) - 2c_2(Y))$, our expression for the Chern numbers of $Y_{k(a,b)}$ obtained in Proposition 5.3 yields the following result:

PROPOSITION 5.6. $\tau(Y_{k(a,b)}) = \frac{(2ab+kb^2)!}{12}\{4(ab - 3a - 3b + 5) + 2k(b-3)b\}$.

In view of the Watershed Conjecture and its role in the "geography of surfaces", as mentioned in the Introduction, the following corollary (0.3 in [**MoRoTe**]) is interesting:

COROLLARY 5.7. *Let $a \geq 1$. Then $\tau(Y_{k(a,b)}) > 0$ if and only if one of the following is true:*

$$\begin{array}{lll} k = 0, & a \geq 8, & b = 4; \\ k = 0, & a \geq 6, & b \geq 5; \\ k = 1, & a \geq 6, & b = 4; \\ k = 1, & a \geq 3, & b = 5; \\ k = 1, & a \geq 2, & b = 6; \\ k = 1, & a \geq 1, & b \geq 7; \\ k = 2, & a \geq 4, & b = 4; \\ k = 2, & a \geq 1, & b \geq 5; \\ k = 3, & a \geq 2, & b = 4; \\ k = 3, & a \geq 1, & b \geq 5; \\ k \geq 4, & a = 1, & b \geq 4. \end{array}$$

From 5.1, 5.6 and 5.7 above, we get the following theorem which appeared in a different form in [**MoRoTe**], Theorem 0.4:

THEOREM 5.8. *Let $a \geq 1$.*

a. *$Y_{k(a,b)}$ is simply connected, of general type, of zero signature, and a spin manifold if one of the following is true:*

(i) $k = 0, \quad a = 7, \quad b = 4;$
(ii) $k = 1, \quad a = 5, \quad b = 4;$
(iii) $k = 2, \quad a = 3, \quad b = 4;$
(iv) $k = 3, \quad a = 1, \quad b = 4.$

b. *There are infinitely many triples k, a, b, for which $Y_{k(a,b)}$ is simply connected, of general type, of positive signature, and a spin manifold. These include infinitely many triples for which $k = 1$ or $k \geq 1$. We present, for example,*

$$k = 1, \qquad a = 3, \qquad b = 5, 6, 7, 8, \ldots$$

§6. Intermediate Galois Covers

As mentioned in the introduction, it was wrongly conjectured that simply connected surfaces of general type only exist in the range $c_1^2/c_2 < 2$ for the "Chern quotient". The first counterexamples, constructed in 1985, had the Chern quotient c_1^2/c_2 just above 2. Those surfaces were $Y_{0(a,b)}$ for certain choices of a, b. This was still rather far away from the maximum value $c_1^2/c_2 = 3$ that follows from the famous inequality of Miyaoka and Yau. Since a surface with $c_1^2/c_2 = 3$ is a free quotient of the unit ball in $\mathbb{C}^2$, it can never have finite fundamental group; in particular, it can never be simply connected. It is thus of interest to find out how close one can get to the quotient $c_1^2/c_2 = 3$.

In order to obtain *spin* simply connected algebraic surfaces with positive signature (as in 5.8) with c_1^2/c_2 closer to 3, we take an intermediate step in the fibered product. We defined *intermediate Galois covers or ℓ-th Galois cover* as the surface obtained from a fibered product taken ℓ times for $\ell < \deg X$. In fact, these constructions give us c_1^2/c_2 closer to 3.

In [**Te2**], Theorem 1, we computed the Chern numbers of the ℓ-th Galois cover in terms of ℓ, $\deg X$, and the following invariants connected to S, the branch curve of $X \to \mathbb{CP}^2$: degree $(= m)$, number of cusps $(= \rho)$, number of nodes $(= d)$, and $\deg S^* = \deg S^{\text{dual}}$ $(= \mu)$. For the branch curve of the Hirzebruch surface, these invariants were computed in [**MoRoTe**] and in [**MoTe2**], and the results are as follows:

LEMMA 6.1. *Let $F_{k(a,b)}$ be as above* (§1).

(i) *If $a \geq 1$, then*
$n = 2ab + kb^2$,
$m = 6ab - 2a - 2b + k(3b^2 - b)$,
$\mu = 6ab - 4a - 4b + 4 + k(3b^2 - 2b)$,
$\varphi = 24ab - 18a - 18b + 12 + k(12b^2 - 9b)$,

(ii) *If $a = 0$, $k = 1$, then*
$n = b^2$,
$m = 3b(b - 1)$,
$\mu = 3(b - 1)^2$,
$\varphi = 3(b - 1)(4b - 5)$,
$d = \dfrac{3}{2}\,(b - 1)(3b^3 - 3b^2 - 14b + 16)$.

PROOF.
(i) [**MoRoTe**], Lemma 7.1.3.
(ii) [**MoTe2**], §2. □

If one substitutes the above $n,\ m,\ \mu,\ d,\ \varphi$ in the formulas of [**Te2**], one gets the Chern classes of the ℓ-th Galois cover of $F_{k(a,b)}$.

Theorem 6.2 treats the case $\ell = n$ (the full Galois cover), and Corollary 6.3 treats an intermediate step $\ell = 4 < 9 = n$.

THEOREM 6.2. *For $k = 1$, $a = 0$ (Veronese surface of order b) we get*

$$c_1^2(Y_{1(0,b)}) = \frac{9}{4}(b^2)!\{b^4 - 2b^3 - 3b^2 + 4b + 4\},$$

$$c_2(Y_{1(0,b)}) = \frac{(b^2)!}{8}\{16 + 54b - 29b^2 - 18b^3 + 9b^4\}.$$

PROOF. [**Te2**], Theorem 1 and Lemma 6.1(b). □

Computing c_1^2/c_2 for $\ell = n$ gives us numbers close to the line $c_1^2 = 2c_2$. On the other hand:

COROLLARY 6.3. *For $k = 1$, $a = 0$, $b = 3$ and $\ell = 4$, we get $c_1^2/c_2 = 2.73$ (almost precisely).*

REMARK. By experimental substitution it seems that for large b, the signature $= \frac{1}{3}(c_1^2 - 2c_2)$ changes from negative to positive around $\ell = \frac{3}{4}n$.

References

[AmTe] Amram M., Teicher M., *Braid monodromy of special projective curves*, in preparation.

[CiMiTe] Ciliberto C., Miranda R., Teicher M., *Braid monodromy of branch curves of K3-surfaces*, in preparation.

[FH] Feustel / Holzapfel, *Symmetry points and Chern invariants of Picard-modular-surfaces*, Math. Nach. **111** (1983), 7-40.

[FRoTe] Freitag P., Robb A., Teicher M., *Finite fundamental groups of Galois covers of rational ruled surfaces*, in preparation.

[H] Hirzebruch F., *Uber eine Klasse von einfach-zusammenhängenden komplexen Mannigfaltigkeiten*, Math. Ann. **124** (1951), 77-86; (first title in his collected papers, Springer Verlag, 1987).

[KMS] Karras A., Magnus W., Solitar D., *Combinatorial Group Theory*, Interscience Publishers, New York, 1966.

[Mo] Moishezon, B., *Stable branch curves and braid monodromies*, Algebraic Geometry, Lecture Notes in Math., vol. 862, 1981, pp. 107-192; (Univ. of Illinois at Chicago Circle, Conference Proceedings, 1980, ed. by A. Libgober and P. Wagreich).

[MoRoTe] Moishezon B., Robb A., Teicher M., *On Galois covers of Hirzebruch surfaces*, Math. Ann. **305** (1996), 493-539.

[MoTe1] Moishezon B., Teicher M., *Simply connected algebraic surfaces with positive index*, Invent. Math. **89** (1987), 601-643.

[MoTe2] Moishezon B., Teicher M., *Galois coverings in the theory of algebraic surfaces*, Proc. of Symp. in Pure Math. **46:1** (1987), 47-65; (in Algebraic Geometry, Bowdin 1985, ed. by J. Bloch et al.)..

[MoTe3] Moishezon B., Teicher M., *Braid group techniques in complex geometry I, Line arrangements in $\mathbb{CP}^2$*, Contemp. Math. **78** (1988), 425-555; (in Braids, Proceedings of the Summer Research Conf., Univ. of California at Santa Cruz, 1986, ed. by J. Birman and A. Libgobe).

[MoTe4] Moishezon B., Teicher M., *Braid group techniques in complex geometry II, From arrangements of lines and conics to cuspidal curves*, Algebraic Geometry, Lecture Notes in Math., vol. 1479, 1990, pp. 131-180; (Chicago Conference Proceedings, 1989).

[MoTe5] Moishezon B., Teicher M., *Braid group techniques in complex geometry III, Projective degeneration of V_3*, Contemp. Math. **162** (1996), 313-332; (Classification of Algebraic Varieties, L'Aquila, 1992; Catanese, Fabrizio and Ciliberto, Ciro (eds.)).

[MoTe6] Moishezon B., Teicher M., *Braid group techniques in complex geometry IV, Braid monodromy of the branch curve S_3 of $V_3 \to \mathbb{CP}^2$ and applications to $\pi_1(\mathbb{CP}^2 - S_3, *)$*, Contemp. Math. **162** (1996), 333-358; (Classification of Algebraic Varieties, L'Aquila, 1992; Catanese, Fabrizio and Ciliberto, Ciro (eds.)).

[MoTe7] Moishezon B., Teicher M., *Braid group techniques in complex geometry V, The fundamental group of a complement of a branch curve of a Veronese generic projection*, Comm. in Analysis and Geometry **4** (1996), no. 11, 1-120.

[MoTe8] Moishezon B., Teicher M., *Finite fundamental groups, free over $\mathbb{Z}/c\mathbb{Z}$, for Galois covers of* $\mathbb{CP}^2$, Math. Ann. **293** (1992), 749-766.

[RoTe] Robb A., Teicher M., *Applications of braid group techniques to the decomposition of moduli spaces, new examples*, Topology and its Applications **78** (1997), 143-151.

[Te1] Teicher M., *Braid groups, algebraic surfaces and fundamental groups of complements of branch curves*, Amer. Math. Soc. Publications PSPM **62** (1997), no. 1, 127-150; (in Algebraic Geometry Santa Cruz 1995, ed. by J. Kollar et al.).

[Te2] Teicher M., *Chern classes of fibered products of surfaces*, Documenta Mathematica **3** (1998), 321-342.

[Te3] Teicher M., *New invariants of surfaces*, Contemp. Math. **231** (1999) (to appear).

[Te4] Teicher M., *The fundamental group of a* $\mathbb{CP}^2$*-complement of a branch curve as an extension of a solvable group by a symmetric group*, Math. Ann. (to appear).

[Te5] Teicher M., *Fundamental groups of complements of branch curves of Hirzebruch surfaces*, in preparation.

[Te6] Teicher M., *On the quotient of the braid group by commutators of transversal half-twists and its group actions*, Topology and its Applications **78** (1997), 153-186.

[VK] Van Kampen E.R., *On the fundamental group of an algebraic curve*, Amer. J. Math. **55** (1933), 255-260.

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